

The (relative) monad-theory correspondence

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2. Theories & relative monads
3. Extending relative monads
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Algebraic theories

Denote by \mathbb{F} the free category with strict finite coproducts on a single object.

An **algebraic theory** is an identity-on-objects functor

$$k : \mathbb{F} \rightarrow \mathcal{B}$$

strictly preserving finite coproducts.

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Sifted-cocontinuous monads

A colimit in Set is **sifted** if it commutes with finite products.

A monad on Set is **sifted-cocontinuous** if it preserves sifted colimits.

These are the same as **finitary monads** on Set .

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Why is there a correspondence between theories and monads?

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theories and monads?

(And why do we care?)

Caring

There are two main reasons we might care about this problem.

- One is **theoretical**: the correspondence is deep and surprising, and understanding it would shed light on a fundamental categorical phenomenon.
- The other is **practical**: the correspondence gives us two perspectives on categorical algebra, each of which is useful in different contexts.

Theories - - - - - Monads

Theories - - Relative monads - - - Monads

What is a relative monad?

Let $j: A \rightarrow E$ be a functor. A j -relative monad¹ consists of

- ▶ a function $t: |A| \rightarrow |E|$;
- ▶ for each $a \in A$, a morphism $\eta_a: ja \rightarrow ta$;
- ▶ for each $a, b \in A$, and morphism $f: ja \rightarrow tb$, a morphism $f^\dagger: ta \rightarrow tb$,

satisfying unitality and associativity laws.

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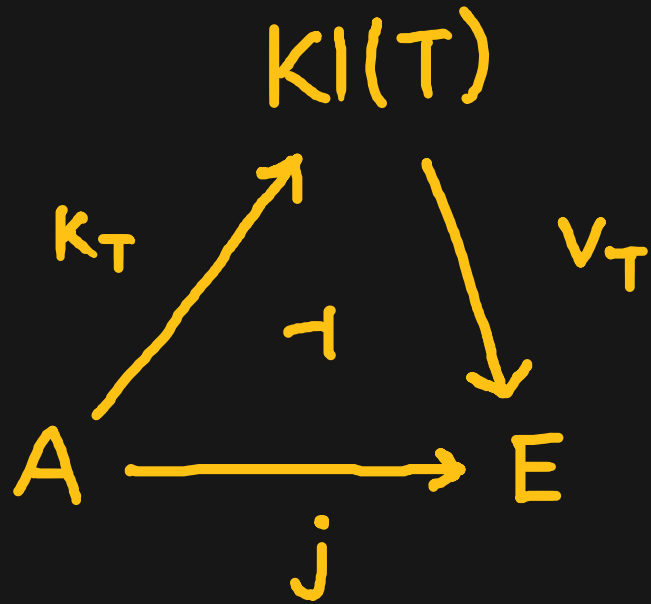
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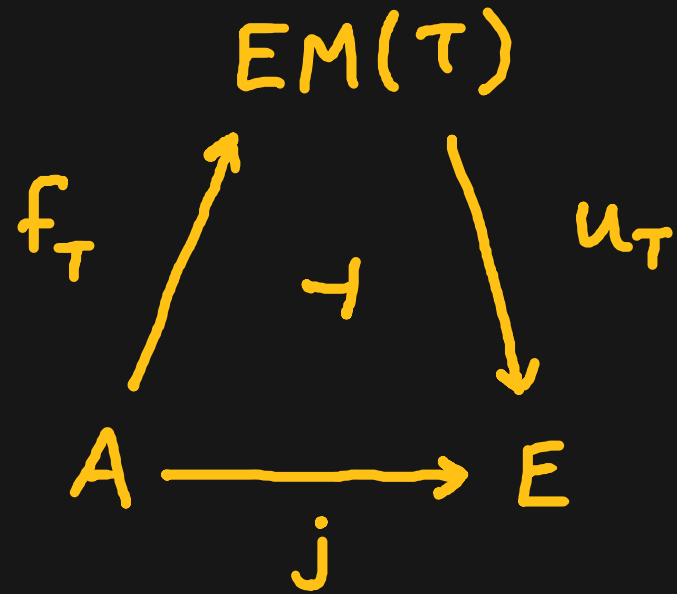
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When j is the identity, this is the same as a monad.

Every relative monad T is induced canonically by two relative adjunctions.¹



Kleisli resolution
(initial)

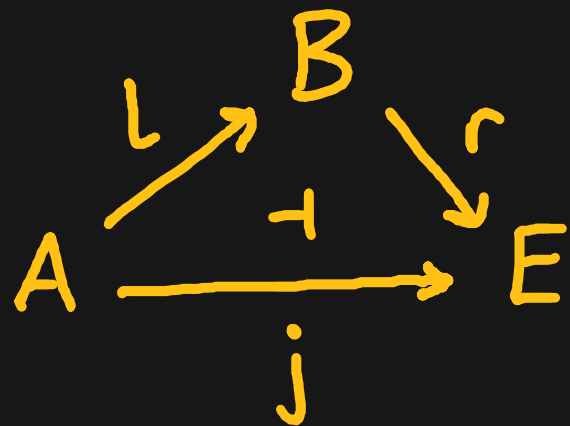


Eilenberg-Moore resolution
(terminal)

What is a relative adjunction?

Let $j: A \rightarrow E$ be a functor. A j -relative adjunction² $L \dashv r$ consists of functors

$L: A \rightarrow B$ and $r: B \rightarrow E$



along with a natural isomorphism

$$B(Lx, y) \cong E(jx, ry)$$

Thus, the following are in bijection:

- ▶ j -relative monads;
- ▶ j -relative Kleisli resolutions;
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A characterisation of Kleisli inclusions

A functor $K: A \rightarrow B$ is the Kleisli inclusion for a j -relative monad if and only if

- ▶ K has a right j -relative adjoint;
- ▶ K is identity-on-objects.

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\mathcal{K} -relative adjunctions

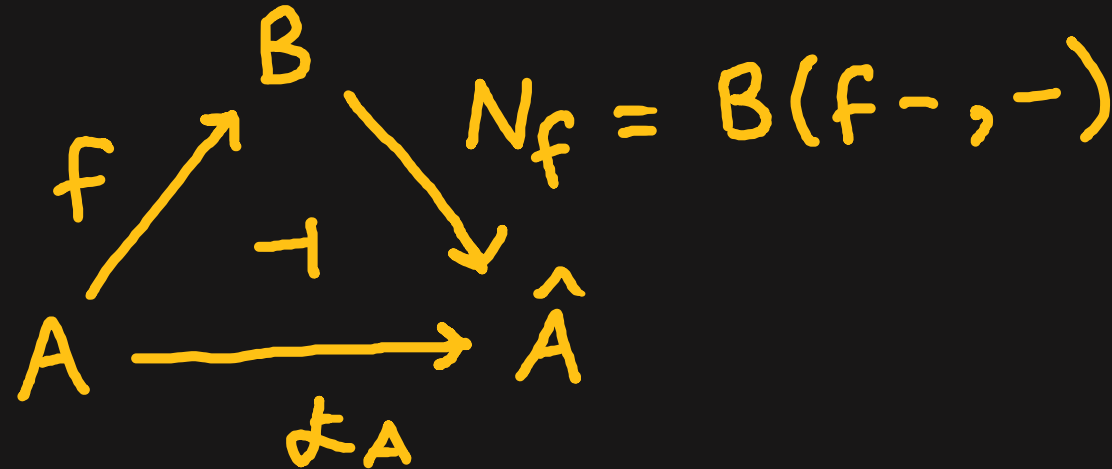
Let $f: A \rightarrow B$ be a functor with small domain.
Then there is a \mathcal{K}_A -relative adjunction

$$\begin{array}{ccc} & B & \\ \begin{array}{c} \nearrow f \\ \dashrightarrow \perp \\ \searrow N_f = B(f-, -) \end{array} & & \\ A & \xrightarrow{\quad \mathcal{K}_A \quad} & \hat{A} \end{array}$$

Hence, every (small) functor is a left \mathcal{K} -relative adjoint.

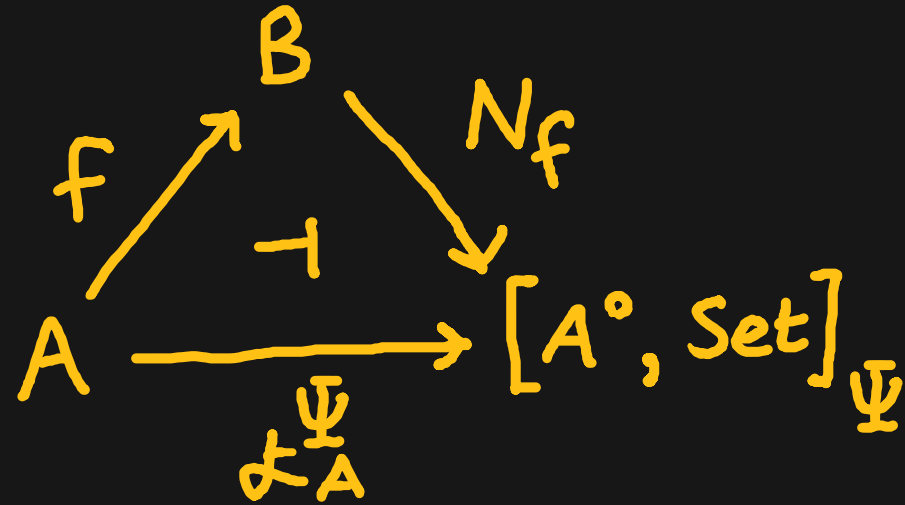
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In particular, every (small) functor preserving finite coproducts is left-adjoint relative to the inclusion into $\text{FinProd}(-^\circ, \text{Set}) \simeq \text{Sind}(-)$.

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Therefore, an algebraic theory is equivalently:

- ▶ an identity-on-objects functor $K: \mathbb{F} \rightarrow \mathbb{B}$ strictly preserving finite coproducts;
- ▶ an identity-on-objects functor $K: \mathbb{F} \rightarrow \mathbb{B}$ which is a left $\mathcal{L}_{\mathbb{F}}^{\text{sift}}$ -relative adjoint;
- ▶ the Kleisli inclusion of a $\mathcal{L}_{\mathbb{F}}^{\text{sift}}$ -relative monad.

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Algebraic theories are hence in bijection with $\mathcal{L}_{\mathbb{F}}^{\text{sift}}$ -relative monads. ($\mathcal{L}_{\mathbb{F}}^{\text{sift}} = \mathbb{F} \longleftrightarrow \text{Set}$)

Theories  — Relative monads - - - Monads
(Kleisli inclusions)

Monads and relative monads

Let Φ be a class of weights. There is an equivalence of categories

$$\mathbf{RMnd}(\mathcal{K}_A^\Phi) \simeq \mathbf{Mnd}_\Phi(\Phi A)$$

$((A \rightarrow \Phi A)$ -relative monads) $(\Phi$ -cocontinuous monads)

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(($A \rightarrow \Phi A$)-relative monads) (Φ -cocontinuous monads)

Hence, $\mathcal{K}_\mathbb{F}^{\text{sift}}$ -relative monads are equivalent to sifted-cocontinuous monads on $\mathbf{Sind}(\mathbb{F}) \simeq \mathbf{Set}$,
i.e. **finitary monads on \mathbf{Set} .**

Theories $\xrightarrow{\text{✓}}$ Relative monads $\xrightarrow{\text{✓}}$ Monads
(Kleisli inclusions) (Extension via cocompletion)

Q

How general is this story?

Theories — Relative monads — Monads

Theories  Relative monads — Monads

(Kleisli inclusions)

Theories $\overset{\textcircled{\checkmark}}{\text{---}}$ Relative monads $\overset{\textcircled{\sim}}{\text{---}}$ Monads
(Kleisli inclusions) (Under suitable assumptions)

Theories

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When j is the inclusion of a category into its cocompletion under a class of weights, we obtain a correspondence between relative monads (and hence theories) and monads.

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But what about more general j ?

j-ary monads and realisable relative monads (1)

Let $j:A \rightarrow E$ be a functor. A monad T on E is **j-ary** if there exists a j -relative monad T' with the same algebras, i.e.

$$\begin{array}{ccc} EM(T) & \xrightarrow{\cong} & EM(T') \\ & \searrow & \swarrow \\ & E & \end{array}$$

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$$\begin{array}{ccc} EM(T) & \xrightarrow{\cong} & EM(T') \\ & \searrow & \swarrow \\ & E & \end{array}$$

A **j-relative monad** T' is **realisable** if there exists a monad T on E with the same algebras.

j-ary monads and realisable relative monads (2)

There is an equivalence between the realisable j-relative monads and the j-ary monads, which commutes with taking categories of algebras.

$$\begin{array}{ccc} \text{RMnd}^E(j) & \xrightarrow{\cong} & \text{Mnd}_j(E) \\ \text{EM} \searrow & \cong & \swarrow \text{EM} \\ & \text{CAT}/E & \end{array}$$

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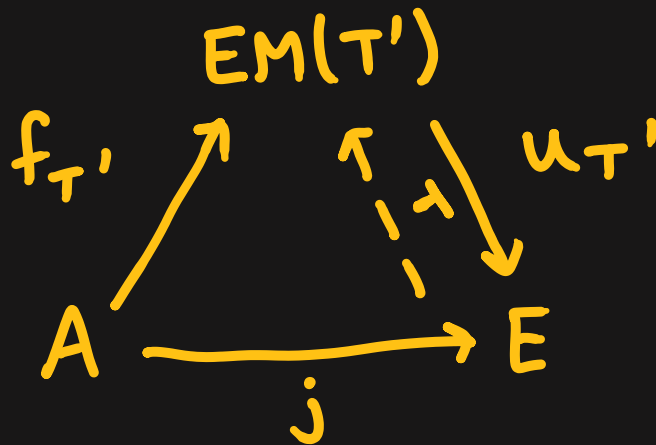
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The functor $\text{Mnd}_j(E) \longrightarrow \text{RMnd}^E(j)$ is given canonically by precomposing j .

j-ary monads and realisable relative monads (3)

In nice settings, we can characterise the **realisable relative monads** and **j-ary monads**.

Prop. Let $j: A \rightarrow E$ be a functor. A j -relative monad T' is realisable if and only if $u_{T'}: EM(T') \rightarrow E$ admits a left adjoint.



j-absolute colimits (1)

A colimit in E is j -absolute if it is preserved by $E \xrightarrow{N_j} \hat{A}$.

$$\begin{array}{ccccc} & & D & & \\ & & \downarrow d & & \\ A & \xrightarrow{j} & E & \longrightarrow & \hat{A} \\ & & N_j = E(j-, -) & & \end{array}$$

That is: $\text{colim}(d); N_j \cong \text{colim}(d; N_j)$.

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When j is the identity, the j -absolute colimits are precisely the absolute colimits, i.e. those preserved by every functor.

j -ary monads and realisable relative monads (4)

In nice settings, we can characterise the realisable relative monads and j -ary monads.

Prop. Let j be admissible and dense, and suppose that left Kan extensions along j exist and are pointwise and j -absolute. Then

(1) Every j -relative monad is realisable.

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(1) Every j -relative monad is realisable.

(2) A monad T is j -ary if and only if

$T \cong \text{Lan}_j(T \circ j)$ if and only if

T preserves j -absolute colimits.

A general* monad-theory correspondence

Theorem. Let j be an admissible, dense functor and suppose that left Kan extensions along j exist and are pointwise and j -absolute.

Then there is an equivalence

$$\text{Th}(j) \simeq \text{Mnd}_j(E)$$

between the categories of j -theories and the monads on E preserving j -absolute colimits.

* We can weaken these assumptions further to obtain an even more general correspondence, but lose the nice characterisations of j -ary monads.

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(However, it is necessary to weaken the assumptions further to compare with some of the existing correspondences, such as that of Bourke-Garner (2019).)

Q

What about the algebras?

Algebras via cocompletion (1)

Theorem. Let $j: A \rightarrow E$ be admissible and dense, and let T be a j -relative monad. The following square forms a pullback.

$$\begin{array}{ccc} EM(T) & \xrightarrow{\quad} & \widehat{KI(T)} \\ u_T \downarrow & \lrcorner & \downarrow k_T^* = [k_T^\circ, \text{Set}] \\ E & \xrightarrow{N_j} & \widehat{A} \end{array}$$

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(When $j = 1_E$, this observation is due to Linton⁶.)

Algebras via cocompletion (2)

Corollary. Let $j: A \rightarrow E$ be admissible and dense, and let $k: A \rightarrow B$ be a j -theory. Denote by T the j -relative monad corresponding to k . The following square forms a pullback.

$$\begin{array}{ccc} EM(T) & \rightarrow & \widehat{B} \\ \downarrow u_T & \lrcorner & \downarrow [k^{\circ p}, \text{Set}] \\ E & \xrightarrow{N_j} & \widehat{A} \end{array}$$

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Algebras via cocompletion (3)

Corollary. Let $L: \mathbb{F} \rightarrow \mathcal{L}$ be an algebraic theory, i.e. a $(\mathbb{F} \leftarrow \text{Set})$ -theory. Denote by T_L the relative monad corresponding to L . The following square forms a pullback.

$$\begin{array}{ccc} EM(T_L) & \xrightarrow{\quad} & \widehat{\mathcal{L}} \\ \downarrow u_T & \lrcorner & \downarrow [L^{\text{op}}, \text{Set}] \\ \text{Set} & \xrightarrow{\quad} & \widehat{\mathbb{F}} \\ & N_{\mathcal{L}} \text{ sind} & \end{array}$$

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Models & algebras

In other words, the models for an algebraic theory coincide with the algebras for the corresponding monad due to the pullback characterisation of the Eilenberg-Moore category for a relative monad.

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- ▶ j -relative Kleisli resolutions;
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Relative monadicity

Theorem. Let $j: A \rightarrow E$ be a dense functor.

TFAE for a functor $r: X \rightarrow E$.

(1) r creates small j -absolute colimits, and has a left j -relative adjoint.

(2) r is j -relatively monadic.

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(This is a Paré-style monadicity theorem³, rather than Beck-style⁴.)

Optheories

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$Opth(j)$ is the full subcategory of CAT/E on the optheories.

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ u \downarrow & & \downarrow u' \\ & E & \end{array}$$

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There is an equivalence

$$Th(j) \simeq RMnd(j) \simeq Opth(j)^{op}$$

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Theories & optheories

In a suitable sense, theories and optheories are dual to one another:

- j -theories axiomatise the **Kleisli inclusions** of j -relative monads (i.e. initial resolutions).
- j -optheories axiomatise the **forgetful Eilenberg - Moore functors** of j -relative monads (i.e. terminal resolutions).

Q

What can we say about the process of passing from theories to optheories and vice versa?

Pre(op)theories

Let $j: A \rightarrow E$ be a dense functor.

An **A-pretheory** is a functor $k: A \rightarrow B$ that is the identity on objects.

An **E-preoptheory** is a functor $u: X \rightarrow E$ that strictly creates small j -absolute colimits.

Pre(op)theories

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(Op)theories are pre(op)theories with relative adjointness properties.

The structure-semantics adjunction (1)

There is an adjunction⁵

$$\begin{array}{ccc} & \xrightarrow{\text{Str}} & \\ (\text{CAT/Set})^* & \perp & \text{AlgTh}^{\text{op}} \\ & \xleftarrow{\text{Sem}} & \end{array}$$

Str (structure) sends a functor $u: X \rightarrow \text{Set}$ to the algebraic theory with operations $\{u^n \Rightarrow u\}_{n \in \mathbb{N}}$.

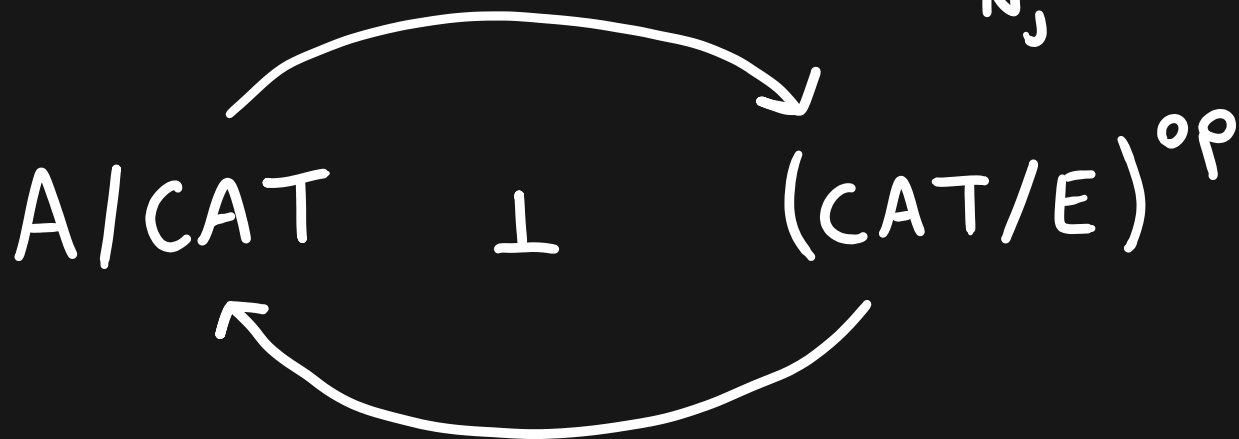
Sem (semantics) sends an algebraic theory $L: \mathbb{F} \rightarrow \mathbb{2}$ to the forgetful functor $\text{Mod}(L) \rightarrow \text{Set}$.

The structure-semantics adjunction (2)

Fix a functor $j: A \rightarrow E$ between small categories.

There is an adjunction⁶

$$k \mapsto \begin{array}{ccc} X & \xrightarrow{\quad} & \hat{B} \\ \downarrow & \lrcorner & \downarrow k' \\ E & \xrightarrow{N_j} & \hat{A} \end{array}$$



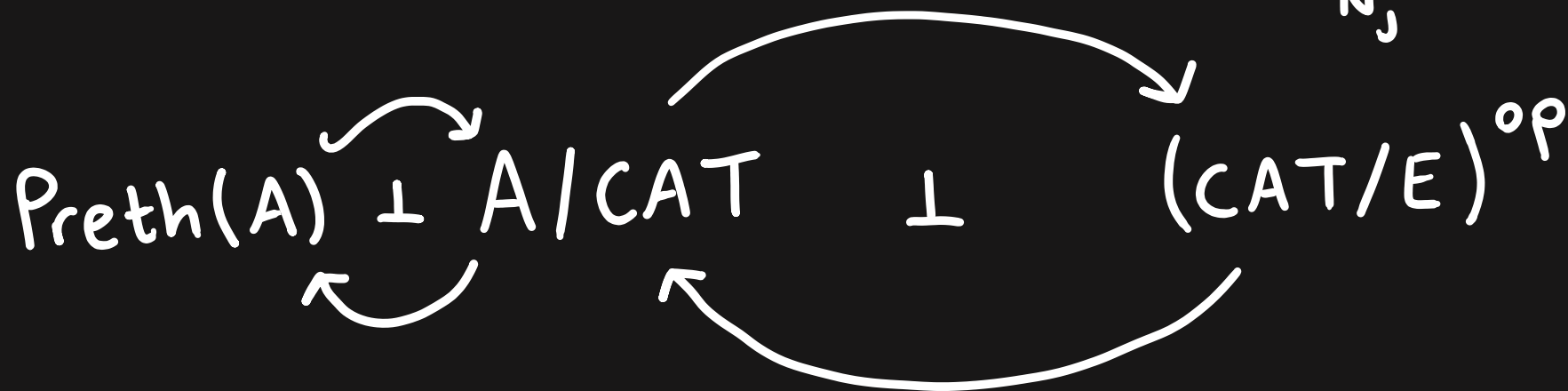
$$u \mapsto A^{op} \xrightarrow{j^{op}} E^{op} \xrightarrow{\iota} \hat{E}^{op} \xrightarrow{(u^{op})^*} \hat{X}^{op}$$

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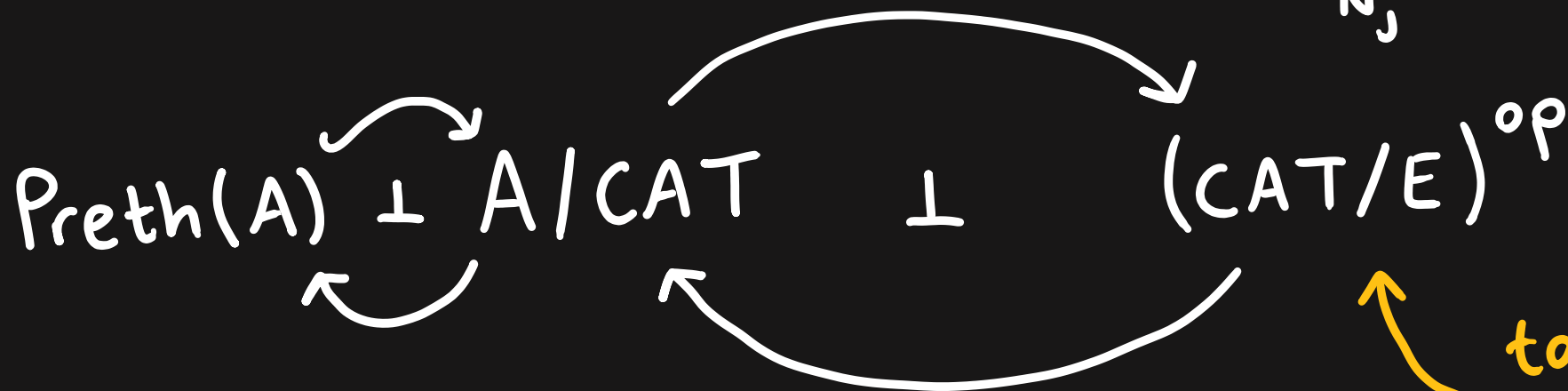
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takes values in preoptheories

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The structure-semantic adjunction (4)

Fix a functor $j: A \rightarrow E$ between small categories.

There is an adjunction

$$\text{Preth}(A) \begin{array}{c} \xrightarrow{\text{Sem}} \\ \perp \\ \xleftarrow{\text{Str}} \end{array} \text{Preopth}(E)^{\text{op}}$$

The structure-semantics adjunction (4)

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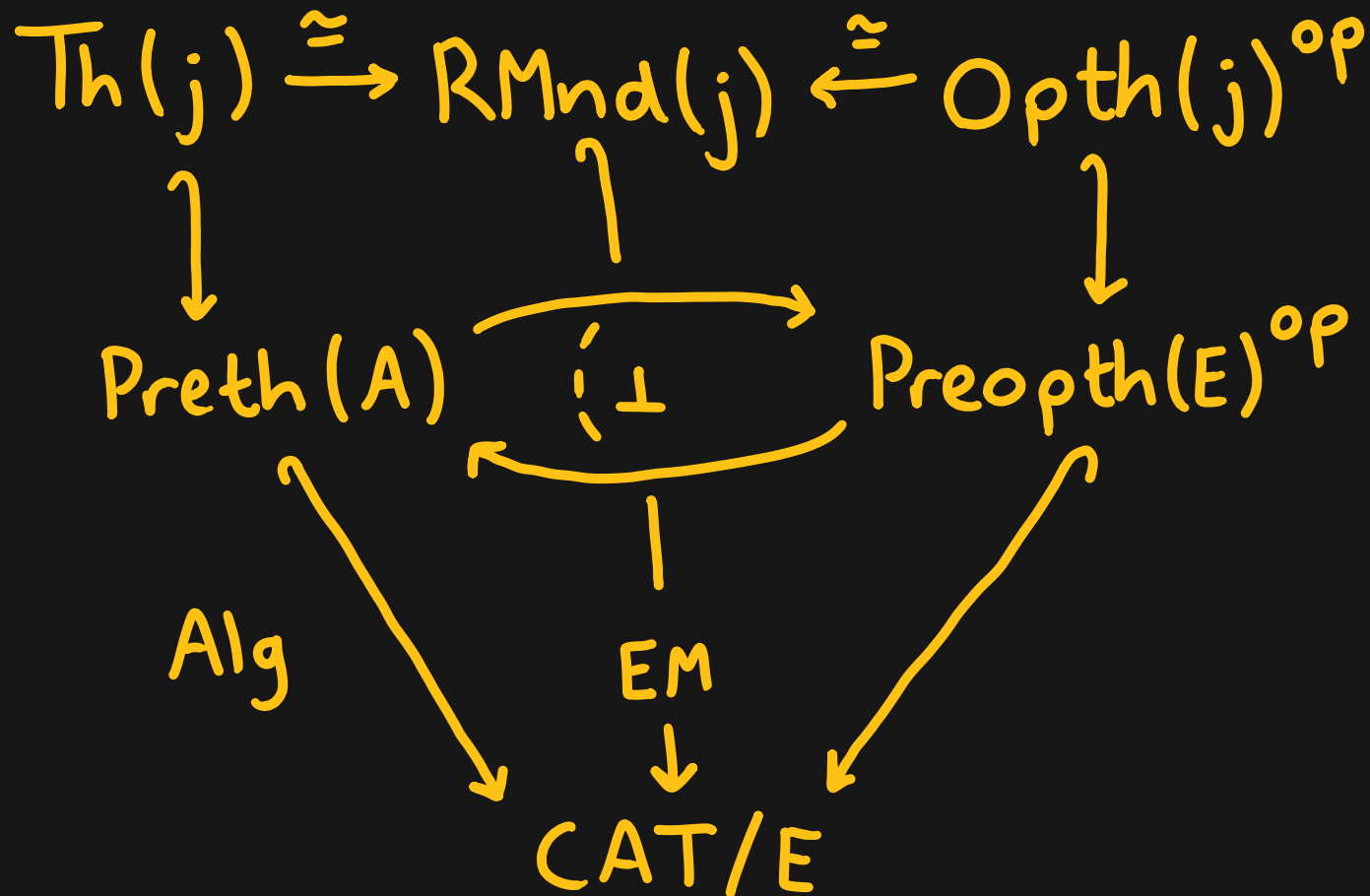
$$\text{Preth}(A) \begin{array}{c} \xrightarrow{\text{Sem}} \\ \perp \\ \xleftarrow{\text{Str}} \end{array} \text{Preopth}(E)^{\text{op}}$$

Furthermore, this adjunction restricts to an adjoint equivalence

$$\text{Th}(j) \simeq \text{Opth}(j)^{\text{op}}$$

The big picture

Fix a dense functor $j: A \rightarrow E$.



Q

This is all good and well,
but is such a general
understanding useful?

Are there new examples?

Enrichment

Our approach is essentially independent of the base of enrichment. For concreteness, we develop our results in the setting of categories enriched in a bicategory \mathcal{W} .

This has two advantages:

1. We capture interesting examples not captured by other frameworks.
2. We subsume other enriched settings.

Examples (1)

Lucyshyn-Wright (2016) develops a monad-theory correspondence with respect to an 'eleutheric system of arities' $j: \mathcal{J} \rightarrow \mathcal{V}$.

Lemma. A \mathcal{J} -theory in the sense of Lucyshyn-Wright is precisely a j -theory in our sense, hence a j -relative monad.

Lucyshyn-Wright's correspondence is thus an instance of ours.

Examples (2)

Bourke-Garner (2019) develop a monad-theory correspondence with respect to a dense, fully faithful functor $j: A \rightarrow E$ with small domain and locally presentable codomain.

Lemma. An A -theory in the sense of Bourke-Garner is precisely a j -theory in our sense, hence a j -relative monad. An A -nervous monad in the sense of Bourke-Garner is precisely a j -ary monad in our sense.

Examples (2)

Bourke-Garner's correspondence is thus an instance of ours.

Examples (3)

- Hoare's (1987) framework for data refinement, as reframed in the setting of enrichment in the nonsymmetric monoidal category LocOrd_l of small locally-ordered categories, locally-ordered functors, and lax natural transformations with a Gray tensor product by Kinoshita-Power (1995).

Examples (3)

- Hoare's (1987) framework for data refinement, as reframed in the setting of enrichment in the nonsymmetric monoidal category LocOrd_l of small locally-ordered categories, locally-ordered functors, and lax natural transformations with a Gray tensor product by Kinoshita-Power (1995).
- The internal monad-theory correspondence of Johnstone-Wraith (1977) via enrichment in the bicategory $\text{Span}(\mathcal{E})$ for a topos \mathcal{E} .

Summary (1)

- The monad-theory correspondence is a combination of two phenomena: the coincidence of theories and relative monads, and the relationship between relative monads and monads with the same algebras.

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- The monad-theory correspondence is a combination of two phenomena: the coincidence of theories and relative monads, and the relationship between relative monads and monads with the same algebras.
- The definition of the category of models for a theory arises from the characterisation of the Eilenberg-Moore category as a pullback over presheaves on the Kleisli category.

Summary (2)

- The structure-semantics adjunction concerns the passage between pretheories and preoptheories, from which the monad-theory(-optheory) correspondence can be seen to arise.

Summary (2)

- The structure-semantics adjunction concerns the passage between pretheories and preoptheories, from which the monad-theory (-optheory) correspondence can be seen to arise.
- This perspective is general enough to encompass all existing (1-dimensional) correspondences.

References

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